On the centralizer of an I-matrix in $M_2(R/I)$, I a principal ideal and R a UFD

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ABSTRACT. The concept of an I-matrix in the full 2×2 matrix ring $M_2(R/I)$, where R is an arbitrary UFD and I is a nonzero ideal in R, was introduced in [**10**]. Moreover a concrete description of the centralizer of an I-matrix $\widehat{B}$ in $M_2(R/I)$ as the sum of two subrings $\mathcal{S}_1$ and $\mathcal{S}_2$ of $M_2(R/I)$ was also given, where $\mathcal{S}_1$ is the image (under the natural epimorphism from $M_2(R)$ to $M_2(R/I)$) of the centralizer in $M_2(R)$ of a pre-image of $\widehat{B}$, and where the entries in $\mathcal{S}_2$ are intersections of certain annihilators of elements arising from the entries of $\widehat{B}$. In the present paper, we obtain results for the case when I is a principal ideal $\langle k\rangle$, $k\in R$ a nonzero nonunit. Mainly we solve two problems. Firstly we find necessary and sufficient conditions for when $\mathcal{S}_1\subseteq\mathcal{S}_2$, for when $\mathcal{S}_2\subseteq\mathcal{S}_1$ and for when $\mathcal{S}_1=\mathcal{S}_2$. Secondly we provide a formula for the number of elements in the centralizer of $\widehat{B}$ for the case when $R/\langle k\rangle$ is finite.

1. Introduction

We denote the centralizer of an element s in an arbitrary ring S by $\mathrm{Cen}_S(s)$. Knowing that $M_n(R)$, the full $n\times n$ matrix ring over a commutative ring R, is a prime example of a non-commutative ring, it is surprising that a concrete description of $\mathrm{Cen}_{M_n(R)}(B)$ for an arbitrary $B\in M_n(R)$ has not yet been found. If $R[x]$ is the polynomial ring in the variable x over R, then

$$\{f(B)\mid f(x)\in R[x]\}\subseteq \mathrm{Cen}_{M_n(R)}(B). \tag{1}$$

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In fact, it is known that (see [**7**])

$$\{f(B) \mid f(x) \in R[x]\} = \mathrm{Cen}_{M_n(R)}(\mathrm{Cen}_{M_n(R)}(B)).$$

The most progress, finding a concrete description of $\mathrm{Cen}_{M_n(R)}(B)$, has been made for the case when the underlying ring R is a field (see [**6**], [**8**], [**9**], [**11**] and [**13**]). The following well-known result in this case provides a necessary and sufficient condition for equality in (1).

Theorem 1.1. *If B is an $n \times n$ matrix over a field F, then*

$$\mathrm{Cen}_{M_n(F)}(B) = \{f(B) \mid f(x) \in F[x]\}$$

if and only if the minimum polynomial of B coincides with the characteristic polynomial of B.

The concept of an I-matrix in the full 2×2 matrix ring $M_2(R/I)$, where R is a UFD and I an ideal in R was introduced in [**10**]. In this paper, unless stated otherwise, we assume that R is a UFD, I is a nonzero ideal in R and $k := \gcd(I) \neq 0$. Let $\theta_I : R \to R/I$ and $\Theta_I : M_2(R) \to M_2(R/I)$ be the natural epimorphism and induced epimorphism respectively. We denote the image $\theta_I(b)$ of $b \in R$ by $\hat{b}_I$ and the image $\Theta_I(B)$ of $B \in M_2(R)$ by $\widehat{B}_I$. However, if there is no ambiguity, then we simply write θ, Θ, $\hat{b}$ and $\widehat{B}$ respectively.

Definition 1.1. *We call a matrix* $\begin{bmatrix} \hat{e}_I & \hat{f}_I \\ \hat{g}_I & \hat{h}_I \end{bmatrix} \in M_2(R/I)$ *an I-matrix if $\langle \hat{e}_I - \hat{h}_I, \hat{f}_I \rangle = \langle \hat{t}_I \rangle$ or $\langle \hat{e}_I - \hat{h}_I, \hat{g}_I \rangle = \langle \hat{t}_I \rangle$ or $\langle \hat{f}_I, \hat{g}_I \rangle = \langle \hat{t}_I \rangle$, where $t|k$.*

If R is a PID, then every matrix in $M_2(R/I)$ is an I-matrix. A concrete description of the centralizer of an I-matrix, as the sum of two subrings of $M_2(R/I)$, was given by the following result in [**10**]:

Theorem 1.2. *Let R be a UFD, I a nonzero ideal in R, and let* $\widehat{B}_I = \begin{bmatrix} \hat{e}_I & \hat{f}_I \\ \hat{g}_I & \hat{h}_I \end{bmatrix} \in M_2(R/I)$ *be an I-matrix, then $\mathrm{Cen}(\widehat{B}) = \mathcal{S}_1 + \mathcal{S}_2$, where*

$$\mathcal{S}_1 = \Theta(\mathrm{Cen}(B)) \quad \text{and} \quad \mathcal{S}_2 = \begin{bmatrix} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e} - \hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e} - \hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{bmatrix}.$$

Unfortunately the concrete description in Theorem 1.2 could not be generalized to $n \times n$-matrices, for $n \geq 3$, in the sense of Proposition 1.2. In [**10**], for R a UFD, a matrix was given for every factor ring R/I with zero divisors and every $n \geq 3$ for which equality in (2) does not hold.

Proposition 1.2. *Let R be a commutative ring and let $B = [b_{ij}] \in M_n(R)$. Then*

$$\Theta(\mathrm{Cen}(B)) + [\mathcal{A}_{ij}] \subseteq \mathrm{Cen}(\widehat{B}), \tag{2}$$

where

$$\mathcal{A}_{ij} = \left(\bigcap_{k,\ k\neq j} \operatorname{ann}(\hat{b}_{jk}) \right) \bigcap \left(\bigcap_{k,\ k\neq i} \operatorname{ann}(\hat{b}_{ki}) \right) \bigcap \operatorname{ann}(\hat{b}_{ii} - \hat{b}_{jj}).$$

Regarding Theorem 1.2, an example was also provided in [**10**], where $\mathcal{S}_1 \not\subseteq \mathcal{S}_2$ and $\mathcal{S}_2 \not\subseteq \mathcal{S}_1$, from which the following questions arise: When is $\mathcal{S}_1 \not\subseteq \mathcal{S}_2$, when is $\mathcal{S}_2 \not\subseteq \mathcal{S}_1$ and when is $\mathcal{S}_1 = \mathcal{S}_2$? In Section 2 this questions will be answered for the case when $I \subset R$ is a principal ideal $\langle k \rangle$ generated by a nonzero nonunit $k \in R$.

The problem of enumerating the number of matrices with given characteristics over a finite ring has been treated extensively in the literature. Formulas have been found, for example, for the number of matrices with a given characteristic polynomial [**12**]; the number of matrices over a finite field that are cyclic [**1**] or symmetric [**4**]; and the number of matrices over the ring of integers $\mathbb{Z}$ modulo m, $\mathbb{Z}_m$, that are nilpotent [**2**]. By using the results in [**3**], some of the above mentioned results, where the matrices over a finite field that satisfy some property are enumerated by rank, can be extended to matrices over certain finite rings that satisfy the property under consideration.

Naturally the question whether it is possible to enumerate the number of matrices in $\mathrm{Cen}_{M_n(R)}(B)$, denoted by $|\mathrm{Cen}_{M_n(R)}(B)|$, when R is a finite commutative ring and $B \in M_n(R)$, arises. Using the fact that the dimension of $\mathrm{Cen}_{M_n(F)}(B)$ is known by the following result, due to Frobenius, the answer is straightforward in the case when R is a finite field F.

Theorem 1.3. *Let $B \in M_n(F)$, and suppose that $f_1, \ldots, f_l \in F[x]$ are the invariant factors of B, where f_i divides f_{i-1}, for $i = 2, \ldots, l$. Then the dimension of $Cen_{M_n(F)}(B)$ is given by*

$$\sum_{i=1}^{l} (2i-1)(\deg f_i).$$

For example, if $n = 2$, then the number of elements in $\mathrm{Cen}_{M_n(F)}(B)$ is $|F|^2$, if B is a nonscalar matrix, and it is $|F|^4$ if B is a scalar matrix. Unfortunately the answer is not that easy in the case when R is a finite ring that is not a field.

In Section 3 we define an equivalence relation on $M_2(R/\langle k \rangle)$ and we use this relation, together with Theorem 1.2, and the results in Section 2, to obtain a formula for the number of matrices in $\mathrm{Cen}_{M_2(R/\langle k \rangle)}(\widehat{B})$ when R is a UFD and $R/\langle k \rangle$ is finite, k is a nonzero nonunit element in R and $\widehat{B} \in M_2(R/\langle k \rangle)$.

2. Containment considerations regarding the centralizer of a $\langle k \rangle$-matrix

In this section we answer the following questions: Regarding Theorem 1.2, when is $\mathcal{S}_1 \not\subseteq \mathcal{S}_2$, when is $\mathcal{S}_2 \not\subseteq \mathcal{S}_1$ and when is $\mathcal{S}_1 = \mathcal{S}_2$?

We need the following preliminary definitions and results in Theorem 2.7, the main result of this section.

Since the minimum polynomial and characteristic polynomial of any 2×2 non-scalar matrix over a field coincide, Theorem 1.1 can be written in the following form for the 2×2 case.

Corollary 2.1. *Let* $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(F)$*,* F *a field. Then*

$$\mathrm{Cen}_{M_2(F)}(B) = \begin{cases} (i)\ M_2(F), \ \text{if } e = h,\ f = 0 \text{ and } g = 0 \text{ (i.e. } B \text{ is a scalar matrix)} \\[2ex] (ii) \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in F \right\}, \ \text{if } e \neq h,\ f = 0 \text{ and } g = 0 \\[2ex] (iii) \left\{ \begin{bmatrix} a & 0 \\ b & a - g^{-1}b(e-h) \end{bmatrix} \middle| a, b \in F \right\}, \ \text{if } f = 0 \text{ and } g \neq 0 \\[2ex] (iv) \left\{ \begin{bmatrix} a & b \\ f^{-1}gb & a - f^{-1}b(e-h) \end{bmatrix} \middle| a, b \in F \right\}, \ \text{if } f \neq 0. \end{cases}$$

The following result, giving a concrete description of the centralizer of a matrix in $M_2(R)$, was proved in [**10**]:

Lemma 2.2. *Let* $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$*,* R *a UFD. Then* $\mathrm{Cen}_{M_2(R)}(B)$

$$= \begin{cases} (i)\ M_2(R), \ \text{if } e = h,\ f = 0 \text{ and } g = 0 \text{ (i.e. } B \text{ is a scalar matrix)} \\[2ex] (ii) \left\{ m^{-1}w \begin{bmatrix} e-h & f \\ g & 0 \end{bmatrix} + vE \middle| v, w \in R \right\}, \ \begin{array}{l} \text{if at least one} \\ \text{of } e-h, f, g \text{ is nonzero,} \end{array} \end{cases}$$

where m^{-1} *is the inverse of* $m := \gcd(e-h, f, g)$ *in the quotient field of* R*.*

The following four results can be easily proved.

Lemma 2.3. *Let* S *be a subring of a ring* T *and let* $s \in S$*. Then*

$$\mathrm{Cen}_S(s) = S \cap \mathrm{Cen}_T(s).$$

Lemma 2.4. *Let* $B \in M_n(R)$*, where* R *is a commutative ring. Then*

$$\mathrm{Cen}_{M_2(R)}(B^T) = (\mathrm{Cen}_{M_2(R)}(B))^T.$$

Lemma 2.5. *Let* R *be a UFD. Suppose* $b, k \in R$*,* k *a nonzero nonunit, and* $\delta = \gcd(b, k)$*. Then*

$$\langle t \rangle = \theta^{-1}(\mathrm{ann}(\hat{b}_{\langle k \rangle})),$$

where $t = \delta^{-1}k \in R$*, with* δ^{-1} *the inverse of* δ *in the quotient field of* R*.*

Lemma 2.6. *Let R be a UFD and let $k, x, y \in R$, then*

$$\operatorname{ann}(\hat{d}) = \operatorname{ann}(\hat{x}) \cap \operatorname{ann}(\hat{y})$$

in $R/\langle k\rangle$, with $\gcd(x,y) = d$.

We are now in a position to prove Theorem 2.6.

Theorem 2.7. *Let R be a UFD, $k = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$ and let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$ be such that $\widehat{B}$ is a $\langle k\rangle$-matrix. Then*

(a)

$$\operatorname{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}) = \Theta_{\langle k\rangle}(\operatorname{Cen}_{M_2(R)}(B)) \tag{3}$$

if and only if B is a scalar matrix or satisfies the following conditions for every i, $i = 1, 2, \ldots, m$:

(i) *p_i is not a divisor of at least one of the elements $e-h$, f and g; pick such an element a, and call the remaining two elements b and c, say.*

(ii) *$\gcd(b,c,k) = 1$ or $\hat{a}_{\langle \gcd(b,c,k)\rangle}$ is invertible in $R/\langle \gcd(b,c,k)\rangle$;*

(b)

$$\operatorname{Cen}(\widehat{B}) = \begin{bmatrix} \operatorname{ann}(\hat{f}) \cap \operatorname{ann}(\hat{g}) & \operatorname{ann}(\hat{g}) \cap \operatorname{ann}(\hat{e}-\hat{h}) \\ \operatorname{ann}(\hat{f}) \cap \operatorname{ann}(\hat{e}-\hat{h}) & \operatorname{ann}(\hat{f}) \cap \operatorname{ann}(\hat{g}) \end{bmatrix} \tag{4}$$

if and only if $\hat{f} = \hat{0}$ and $\hat{g} = \hat{0}$;

(c)

$$\Theta(\operatorname{Cen}(B)) = \begin{bmatrix} \operatorname{ann}(\hat{f}) \cap \operatorname{ann}(\hat{g}) & \operatorname{ann}(\hat{g}) \cap \operatorname{ann}(\hat{e}-\hat{h}) \\ \operatorname{ann}(\hat{f}) \cap \operatorname{ann}(\hat{e}-\hat{h}) & \operatorname{ann}(\hat{f}) \cap \operatorname{ann}(\hat{g}) \end{bmatrix} \tag{5}$$

if and only if $\hat{f} = \hat{0}$, $\hat{g} = \hat{0}$ and ($\hat{e}-\hat{h}$ is invertible or $\hat{e}-\hat{h} = \hat{0}$).

PROOF. (a) Since (3) follows trivially if B is a scalar matrix, we assume that B is a non-scalar matrix. Suppose that conditions (i) and (ii) are satisfied. If

$$\operatorname{ann}_{M_2(R/\langle k\rangle)}(\theta_{\langle k\rangle}(\gcd(f,g))) = \hat{0}_{\langle k\rangle}, \qquad \operatorname{ann}_{M_2(R/\langle k\rangle)}(\theta_{\langle k\rangle}(\gcd(f,e-h))) = \hat{0}_{\langle k\rangle} \tag{6}$$

$$\text{and} \qquad \operatorname{ann}_{M_2(R/\langle k\rangle)}(\theta_{\langle k\rangle}(\gcd(g,e-h))) = \hat{0}_{\langle k\rangle}, \tag{7}$$

then the result follows from Theorem 1.2 and Lemma 2.6. Thus suppose that at least one of the annihilators in (6) and (7) is nonzero. We now show that

$$\begin{bmatrix} \hat{0}_{\langle k\rangle} & \operatorname{ann}(\theta_{\langle k\rangle}(\gcd(g,e-h))) \\ \hat{0}_{\langle k\rangle} & \hat{0}_{\langle k\rangle} \end{bmatrix}, \begin{bmatrix} \hat{0}_{\langle k\rangle} & \hat{0}_{\langle k\rangle} \\ \operatorname{ann}(\theta_{\langle k\rangle}(\gcd(f,e-h))) & \hat{0}_{\langle k\rangle} \end{bmatrix},$$

$$\begin{bmatrix} \hat{0}_{\langle k\rangle} & \hat{0}_{\langle k\rangle} \\ \hat{0}_{\langle k\rangle} & \operatorname{ann}(\theta_{\langle k\rangle}(\gcd(f,g))) \end{bmatrix} \in \Theta_{\langle k\rangle}(\operatorname{Cen}_{M_2(R)}(B)). \tag{8}$$

Since then, because $\Theta_{\langle k\rangle}(\mathrm{Cen}_{M_2(R)}(B))$ is a ring, (3) follows from Theorem 1.2 and Lemma 2.6.

If $\mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_{\langle k\rangle}(\gcd(g,e-h)))\neq\hat{0}_{\langle k\rangle}$, then, by Lemma 2.5, $1\neq\gcd(g,e-h,k):=\delta$ and $\mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_{\langle k\rangle}(\gcd(e-h,g)))=\langle(\widehat{k\delta^{-1}})_{\langle k\rangle}\rangle$. To accomplish our objective, we show that for each $\hat{d}_{\langle k\rangle}\in\mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_{\langle k\rangle}(\gcd(g,e-h)))$ there is a $\hat{d}'_{\langle k\rangle}\in\mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_{\langle k\rangle}(\gcd(g,e-h)))$ such that $\hat{f}_{\langle k\rangle}\hat{d}'_{\langle k\rangle}=\hat{d}_{\langle k\rangle}$, since then

$$\Theta_{\langle k\rangle}\left(\begin{bmatrix} 0 & fd' \\ gd' & (e-h)d' \end{bmatrix}\right)=\begin{bmatrix} \hat{0}_{\langle k\rangle} & \hat{d}_{\langle k\rangle} \\ \hat{0}_{\langle k\rangle} & \hat{0}_{\langle k\rangle} \end{bmatrix},$$

so that we therefore can conclude from Lemma 2.2(ii) that

$$\begin{bmatrix} \hat{0}_{\langle k\rangle} & \mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_{\langle k\rangle}(\gcd(g,e-h))) \\ \hat{0}_{\langle k\rangle} & \hat{0}_{\langle k\rangle} \end{bmatrix}\in\Theta_{\langle k\rangle}(\mathrm{Cen}_{M_2(R)}(B)).$$

Thus, let $\hat{d}_{\langle k\rangle}$ be an arbitrary element in $\mathrm{ann}_{M_2(R/\langle k\rangle)}(\theta_{\langle k\rangle}(\gcd(g,e-h)))$, i.e. suppose $\hat{d}_{\langle k\rangle}:=\hat{s}_{\langle k\rangle}(\widehat{k\delta^{-1}})_{\langle k\rangle}$ for some $\hat{s}_{\langle k\rangle}\in R/\langle k\rangle$. Since $\hat{f}_{\langle\delta\rangle}$ is invertible in $R/\langle\delta\rangle$, by (ii), there is a $\hat{t}_{\langle\delta\rangle}\in R/\langle\delta\rangle$ such that $\hat{t}_{\langle\delta\rangle}\hat{f}_{\langle\delta\rangle}=\hat{1}_{\langle\delta\rangle}$ which implies that $tf=1+v\delta$ for some $v\in R$. Hence $ftd=(1+v\delta)(sk\delta^{-1}+wk)=sk\delta^{-1}+(w+vs+v\delta w)k$. In other words, if we set $\hat{d}'_{\langle k\rangle}:=(\widehat{td})_{\langle k\rangle}$ then $\hat{f}_{\langle k\rangle}\hat{d}'_{\langle k\rangle}=\hat{f}_{\langle k\rangle}(\widehat{td})_{\langle k\rangle}=(\widehat{sk\delta^{-1}})_{\langle k\rangle}=\hat{d}_{\langle k\rangle}$.

It can similarly be shown that each of the other two sets in (8) is contained in $\Theta_{\langle k\rangle}(\mathrm{Cen}_{M_2(R)}(B))$.

Conversely, suppose B does not satisfy both of the conditions (i) and (ii). We distinguish between the following cases:

(a′) B does not satisfy (i), i.e. $\gcd(e-h,f,g,k)\neq 1$;
(b′) B satisfies (i), but not (ii).

(a′) Suppose there is a prime p_i in the prime factorization of k such that $p_i|e-h,f,g$. We distinguish between the following two cases:

(i′) $f=0$ or $g=0$;
(ii′) $f,g\neq 0$.

(i′) Since $p_i|e-h,f,g$, direct verification shows that

$$\begin{aligned}\widehat{A}_{\langle k\rangle} &:= \begin{bmatrix} \hat{0} & \theta(p_1^{n_1}\cdots p_{i-1}^{n_{i-1}}p_i^{n_i-1}p_{i+1}^{n_{i+1}}\cdots p_m^{n_m}) \\ \theta(p_1^{n_1}\cdots p_{i-1}^{n_{i-1}}p_i^{n_i-1}p_{i+1}^{n_{i+1}}\cdots p_m^{n_m}) & \hat{0} \end{bmatrix}\\ &\in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}).\end{aligned}$$

Because $\theta_{\langle k\rangle}(p_1^{n_1}\cdots p_{i-1}^{n_{i-1}}p_i^{n_i-1}p_{i+1}^{n_{i+1}}\cdots p_m^{n_m})\neq\hat{0}_{\langle k\rangle}$, it follows that the entries in position $(1,2)$ and position $(2,1)$ of $\widehat{A}_{\langle k\rangle}$ only have nonzero pre-images in R. Since B is a non-scalar matrix, it follows from Lemma 2.2(ii) that every matrix in $\mathrm{Cen}_{M_2(R)}(B)$

has 0 in position $(1,2)$ if $f = 0$ and 0 in position $(2,1)$ if $g = 0$. Therefore $\widehat{A}_{\langle k\rangle} \notin \Theta_{\langle k\rangle}(\mathrm{Cen}_{M_2(R)}(B))$ if $f = 0$ or $g = 0$.

(ii′) Since $f, g \neq 0$ and $p_i | f, g$ it follows that

$$f = cp_i^r \qquad \text{and} \qquad g = dp_i^s \tag{9}$$

for some $s, r \geq 1$ and $c, d \in R$ such that $p_i \nmid c, d$. Now, $r \leq s$ or $s \leq r$. Let us first assume that $r \leq s$. Because $p_i | e - h, f, g$ direct verification shows that

$$\widehat{A}_{\langle k\rangle} := \begin{bmatrix} \hat{0}_{\langle k\rangle} & \hat{0}_{\langle k\rangle} \\ \theta_{\langle k\rangle}(p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_i^{n_i - 1} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m}) & \hat{0}_{\langle k\rangle} \end{bmatrix} \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}).$$

We now show that $\widehat{A}_{\langle k\rangle} \notin \Theta_{\langle k\rangle}(\mathrm{Cen}_{M_2(R)}(B))$. Firstly note that the set of all the pre-images of $\widehat{A}_{\langle k\rangle}$ is

$$\begin{bmatrix} \ker \theta_{\langle k\rangle} & \ker \theta_{\langle k\rangle} \\ p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_i^{n_i - 1} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m} + \ker \theta_{\langle k\rangle} & \ker \theta_{\langle k\rangle} \end{bmatrix}.$$

Thus, if $\widehat{A}_{\langle k\rangle} \in \Theta_{\langle k\rangle}(\mathrm{Cen}_{M_2(R)}(B))$, then, according to Corollary 2.1(iv) and Lemma 2.3 there is a pre-image $\begin{bmatrix} \kappa_1 & \kappa_2 \\ p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_i^{n_i - 1} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m} + \kappa_3 & \kappa_4 \end{bmatrix} \in M_2(R)$ of $\widehat{A}_{\langle k\rangle}$, where $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \ker \theta_{\langle k\rangle}$, such that

$$\begin{bmatrix} \kappa_1 & \kappa_2 \\ p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_i^{n_i - 1} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m} + \kappa_3 & \kappa_4 \end{bmatrix} = \begin{bmatrix} a & b \\ gf^{-1}b & a - (e-h)f^{-1}b \end{bmatrix}$$

in $M_2(R)$ for some $a, b \in R$. In other words, there are $a, b \in R$ such that $\kappa_1 = a$, $\kappa_2 = b$ and $p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_i^{n_i - 1} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m} + \kappa_3 = gf^{-1}b$. But then, considering (9) and keeping in mind that $r \leq s$, $gf^{-1}b \in R$ and $p_i \nmid c, d$, we have that $gf^{-1}b = dp_i^s (cp_i^r)^{-1} \kappa_2 \in \langle p_i^{n_i} \rangle$, where $\langle p_i^{n_i} \rangle$ is the ideal generated by $p_i^{n_i}$ in R. Because $p_i^{n_i} \nmid p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_i^{n_i - 1} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m} + \kappa_3$, it follows that $p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_i^{n_i - 1} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m} + \kappa_3 \notin \langle p_i^{n_i} \rangle$, which implies that

$$p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_i^{n_i - 1} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m} + \kappa_3 \neq gf^{-1}b.$$

Thus we have a contradiction. Therefore $\widehat{A}_{\langle k\rangle} \notin \Theta_{\langle k\rangle}(\mathrm{Cen}_{M_2(R)}(B))$.

If $s \leq r$ one can similarly show that

$$\widehat{A}_{\langle k\rangle} := \begin{bmatrix} \hat{0}_{\langle k\rangle} & \theta_{\langle k\rangle}(p_1^{n_1} \cdots p_{i-1}^{n_{i-1}} p_i^{n_i - 1} p_{i+1}^{n_{i+1}} \cdots p_m^{n_m}) \\ \hat{0}_{\langle k\rangle} & \hat{0}_{\langle k\rangle} \end{bmatrix} \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}),$$

and that $\widehat{A}_{\langle k\rangle} \notin \Theta_{\langle k\rangle}(\mathrm{Cen}_{M_2(R)}(B))$, by using Lemma 2.4 and Corollary 2.1(iv) instead of Corollary 2.1(iv).

(b′) Suppose B satisfies (i), but not (ii). Then, for each $i \in \{1, \ldots, m\}$, at least one of the following cases is true:

(i′) $\gcd(e-h,f,g,k)=1$, $1\neq \gcd(e-h,g,k):=\delta$ and $\hat{f}_{\langle\delta\rangle}$ is not invertible in $R/\langle\delta\rangle$;

(ii′) $\gcd(e-h,f,g,k)=1$, $1\neq \gcd(e-h,f,k):=\delta$ and $\hat{g}_{\langle\delta\rangle}$ is not invertible in $R/\langle\delta\rangle$;

(iii′) $\gcd(e-h,f,g,k)=1$, $1\neq \gcd(f,g,k):=\delta$ and $\hat{e}_{\langle\delta\rangle}-\hat{h}_{\langle\delta\rangle}$ is not invertible in $R/\langle\delta\rangle$;

We now show that (3) does not follow in each of the above cases.

(i′) In this case Lemma 2.5 implies that

$$\operatorname{ann}_{M_2(R/\langle k\rangle)}(\theta_{\langle k\rangle}(\gcd(g,e-h)))=\langle\widehat{(k\delta^{-1})}_{\langle k\rangle}\rangle.$$

Note that since δ is not a unit, $\langle k\delta^{-1}\rangle\neq\langle k\rangle$. By Theorem 1.2 it follows that

$$\widehat{A}_{\langle k\rangle}:=\begin{bmatrix}\hat{0}_{\langle k\rangle} & \widehat{(k\delta^{-1})}_{\langle k\rangle}\\ \hat{0}_{\langle k\rangle} & \hat{0}_{\langle k\rangle}\end{bmatrix}\in \operatorname{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}).$$

If we can show that $\widehat{A}_{\langle k\rangle}\notin\Theta_{\langle k\rangle}(\operatorname{Cen}_{M_2(R)}(B))$, then we are finished. Now,

$$\begin{bmatrix}\ker\theta_{\langle k\rangle} & k^{-1}\delta+\ker\theta_{\langle k\rangle}\\ \ker\theta_{\langle k\rangle} & \ker\theta_{\langle k\rangle}\end{bmatrix}$$

is the set of all the pre-images of $\widehat{A}_{\langle k\rangle}$ in R. Therefore, taking into account that $\gcd(e-h,f,g,k)=1$, if $\widehat{A}_{\langle k\rangle}\in\Theta_{\langle k\rangle}(\operatorname{Cen}_{M_2(R)}(B))$ it follows from Corollary 2.2(ii) that there is a pre-image $\begin{bmatrix}\kappa_1 & k\delta^{-1}+\kappa_2\\ \kappa_3 & \kappa_4\end{bmatrix}\in M_2(R)$ of $\widehat{A}_{\langle k\rangle}$, where $\kappa_1,\kappa_2,\kappa_3,\kappa_4\in\ker\theta_{\langle k\rangle}$, such that

$$\begin{bmatrix}\kappa_1 & k\delta^{-1}+\kappa_2\\ \kappa_3 & \kappa_4\end{bmatrix}=\begin{bmatrix}a & fb\\ gb & a-(e-h)b\end{bmatrix}$$

for some $a,b\in R$. Hence, $gb=\kappa_3$ and $(e-h)b=\kappa_1-\kappa_4$, which implies that $b=sk\delta^{-1}$ for some $s\in R$. But then, since $fb=k\delta^{-1}+\kappa_2$, we have that

$$fb=fsk\delta^{-1}=k\delta^{-1}+\kappa_2\Leftrightarrow fs=1+t\delta\text{ for some }t\in R\Leftrightarrow \hat{f}_{\langle\delta\rangle}\hat{s}_{\langle\delta\rangle}=\hat{1}_{\langle\delta\rangle}.$$

Since $\hat{f}_{\langle\delta\rangle}$ is not invertible in $R/\langle\delta\rangle$, according to assumption, we have a contradiction. Therefore $\widehat{A}_{\langle k\rangle}\notin\Theta_{\langle k\rangle}(\operatorname{Cen}_{M_2(R)}(B))$ and so we conclude that

$$\operatorname{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle})\not\subseteq\Theta_{\langle k\rangle}(\operatorname{Cen}_{M_2(R)}(B)).$$

(ii′ and iii′) It follows similarly to case (i′) that $\operatorname{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B})\not\subseteq\Theta(\operatorname{Cen}_{M_2(R)}(B))$.

(b) Suppose $\hat{f}, \hat{g} = \hat{0}$, then $f, g \in \langle k \rangle$, and so by Corollary 2.2(ii)

$$\begin{aligned}
\Theta(\mathrm{Cen}(B)) &= \Theta\left(\left\{\left[\begin{array}{cc} a & fb \\ gb & a-(e-h)b \end{array}\right] \middle| a, b \in R\right\}\right) \\
&= \Theta\left(\left\{\left[\begin{array}{cc} a & 0 \\ 0 & a-(e-h)b \end{array}\right] \middle| a, b \in R\right\}\right) \\
&\subseteq \left[\begin{array}{cc} R/\langle k \rangle & \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{e}-\hat{h}) & R/\langle k \rangle \end{array}\right] \\
&= \left[\begin{array}{cc} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{array}\right].
\end{aligned}$$

Conversely, suppose $\Theta(\mathrm{Cen}(B)) \subseteq \left[\begin{array}{cc} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{array}\right]$.
Since $\left[\begin{array}{cc} \hat{a} & \hat{0} \\ \hat{0} & \hat{a} \end{array}\right] \in \Theta(\mathrm{Cen}(B))$ for every $\hat{a} \in R/\langle k \rangle$ it follows that $\mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) = R/\langle k \rangle$ which implies that $\mathrm{ann}(\hat{f}) = R/\langle k \rangle$ and $\mathrm{ann}(\hat{g}) = R/\langle k \rangle$ and so $\hat{f}, \hat{g} = \hat{0}$.

(c) Using (b) and (a), it follows that

$$\begin{aligned}
&\Theta(\mathrm{Cen}(B)) = \left[\begin{array}{cc} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{array}\right] \\
\Leftrightarrow\ &\Theta(\mathrm{Cen}(B)) \subseteq \left[\begin{array}{cc} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{array}\right] \text{ and} \\
&\left[\begin{array}{cc} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{array}\right] \subseteq \Theta(\mathrm{Cen}(B)) \\
\Leftrightarrow\ &\hat{f}, \hat{g} = \hat{0} \text{ and } \left[\begin{array}{cc} \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g}) \cap \mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f}) \cap \mathrm{ann}(\hat{g}) \end{array}\right] \subseteq \Theta(\mathrm{Cen}(B)) \\
\Leftrightarrow\ &\hat{f}, \hat{g} = \hat{0} \text{ and } (\hat{e}-\hat{h} \text{ is invertible in } R/\langle k \rangle \text{ or } \hat{e}-\hat{h} = \hat{0}).
\end{aligned}$$

□

Example 2.8. Let $R = F[x, y, z]$, $k = x^3y^2z$ and let $B = \left[\begin{array}{cc} x^2y^2 & x+1 \\ x^2 & 0 \end{array}\right]$, $B' = \left[\begin{array}{cc} x^2y^2 & 0 \\ 0 & 0 \end{array}\right]$ and $B'' = \left[\begin{array}{cc} 1+xyz & 0 \\ 0 & 0 \end{array}\right]$. Note that $\widehat{B}$, $\widehat{B}'$ and $\widehat{B}''$ are $\langle x^3y^2z \rangle$-matrices. Since $\gcd(x^2y^2, x^2) = x^2$ and $\widehat{(x+1)}_{\langle x^2 \rangle}$ is invertible in $R/\langle x^2 \rangle$, it follows from Lemma 2.2(ii) and Theorem 2.7(a) that

$$\mathrm{Cen}(\widehat{B}_{\langle k \rangle}) = \Theta(\mathrm{Cen}(B)) = \left\{\left[\begin{array}{cc} \hat{a} & \widehat{(x+1)}\hat{b} \\ \widehat{x^2}\hat{b} & \hat{a} + \widehat{x^2y^2}\hat{b} \end{array}\right] \middle| \hat{a}, \hat{b} \in F[x, y, z]/\langle x^3y^2z \rangle\right\}.$$

Furthermore, it follows from Theorem 2.7(b) that

$$\mathrm{Cen}(\widehat{B}'_{\langle k\rangle}) = \begin{bmatrix} R/\langle x^3y^2z\rangle & \langle\widehat{xz}\rangle \\ \langle\widehat{xz}\rangle & R/\langle x^3y^2z\rangle \end{bmatrix}$$

and, since $\theta_{\langle x^3y^2z\rangle}(1+xyz)$ is invertible in $R/\langle x^3y^2z\rangle$, from Theorem 2.7(c) that

$$\begin{aligned} \mathrm{Cen}(\widehat{B}'') &= \Theta(\mathrm{Cen}(B''_{\langle k\rangle})) = \begin{bmatrix} \mathrm{ann}(\hat{f})\cap\mathrm{ann}(\hat{g}) & \mathrm{ann}(\hat{g})\cap\mathrm{ann}(\hat{e}-\hat{h}) \\ \mathrm{ann}(\hat{f})\cap\mathrm{ann}(\hat{e}-\hat{h}) & \mathrm{ann}(\hat{f})\cap\mathrm{ann}(\hat{g}) \end{bmatrix} \\ &= \begin{bmatrix} R/\langle x^3y^2z\rangle & \hat{0} \\ \hat{0} & R/\langle x^3y^2z\rangle \end{bmatrix}. \end{aligned}$$

The following result is well-known.

Lemma 2.9. *Let R be a PID. Then an element $\hat{b}\in R/\langle k\rangle$ is invertible if and only if* $\gcd(b,k)=1$.

Using Lemma 2.9 and the fact that every matrix in $M_2(R)$ is a $\langle k\rangle$-matrix if R is a PID, we simplify Theorem 2.7(a) for the case when R is a PID.

Corollary 2.10. *Let R be a PID and let $B=\begin{bmatrix} e & f \\ g & h\end{bmatrix}\in M_2(R)$. Then*

$$\mathrm{Cen}(\widehat{B}) = \Theta(\mathrm{Cen}(B))$$

if and only if B is a scalar matrix or $\gcd(e-h,f,g,k)=1$.

Note that although Corollary 2.11 is not a characterization of the $\langle k\rangle$-matrices for which (3) is true, it is easier to verify if Corollary 2.11 applies to a specific matrix in $M_2(R)$ than to verify if Theorem 2.7(a) applies to a specific matrix in $M_2(R)$.

Corollary 2.11. *Let R be a UFD, $k\in R$ and $B=\begin{bmatrix} e & f \\ g & h\end{bmatrix}\in M_2(R)$. If at least one of the three elements $\hat{e}-\hat{h}$, $\hat{f}$ and $\hat{g}$ is invertible in $R/\langle k\rangle$, then*

$$\mathrm{Cen}(\widehat{B}) = \Theta(\mathrm{Cen}(B)).$$

Proof. It follows trivially that $\widehat{B}$ is a $\langle k\rangle$-matrix. Without loss of generality, let us suppose that $\hat{f}$ is invertible in $R/\langle k\rangle$. Then, by Lemma 2.9 $\gcd(f,k)=1$. Hence condition (i) in Theorem 2.7(a) is satisfied. Now, suppose that $\gcd(e-h,g,k)=\delta$. If δ is a unit, then condition (ii) is also satisfied. Thus suppose that δ is not a unit. Then, since $\hat{f}_{\langle k\rangle}$ is invertible in $R/\langle k\rangle$ and $\delta|k$, it follows that there is a $t\in R$ such that $tf=1+sk=1+sv\delta$ for some $s,v\in R$ which implies that $\hat{t}_{\langle\delta\rangle}\hat{f}_{\langle\delta\rangle}=\hat{1}_{\langle\delta\rangle}$. Therefore condition (ii) in Theorem 2.7(a) is also satisfied. □

3. The number of matrices in the centralizer of a matrix in $M_2(R/\langle k\rangle)$, R a UFD and $R/\langle k\rangle$ finite

In this section $k \in R$ will always be a nonzero nonunit such that $R/\langle k\rangle$ is finite and we will always denote the number of elements in a ring S by $|S|$. The purpose of this section is to determine the number of matrices in $\mathrm{Cen}_{M_2(R/\langle k\rangle)}(B)$, where R is a UFD, $R/\langle k\rangle$ is finite and $B \in M_2(R/\langle k\rangle)$.

To reach our goal, we first need some preliminary results.

Definition 3.1. *Let $k \in R$, let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$ and let $d := \gcd(e-h, f, g, k)$. We define the relation $\sim$ on $\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle})$ as follows: for $\widehat{A}_{\langle k\rangle}, \widehat{C}_{\langle k\rangle} \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle})$,*

$$\widehat{A}_{\langle k\rangle} \sim \widehat{C}_{\langle k\rangle} \qquad \textit{iff} \qquad \widehat{A}_{\langle k\rangle} - \widehat{C}_{\langle k\rangle} \in M_2(\langle \widehat{(kd^{-1})}_{\langle k\rangle}\rangle).$$

It follows immediately that $\sim$ is an equivalence relation.
We denote the equivalence class of $\widehat{A}_{\langle k\rangle}$ by $\widehat{A}^*_{\langle k\rangle}$ and the set

$$\{\widehat{A}^*_{\langle k\rangle} \mid \widehat{A}_{\langle k\rangle} \in (\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}))\}$$

of all equivalence classes by

$$(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}))^*.$$

Since

$$M_2(\langle \widehat{(kd^{-1})}_{\langle k\rangle}\rangle) \subseteq \begin{bmatrix} \mathrm{ann}(\hat{f}_{\langle k\rangle}) \cap \mathrm{ann}(\hat{g}_{\langle k\rangle}) & \mathrm{ann}(\hat{e}_{\langle k\rangle} - \hat{h}_{\langle k\rangle}) \cap \mathrm{ann}(\hat{g}_{\langle k\rangle}) \\ \mathrm{ann}(\hat{e}_{\langle k\rangle} - \hat{h}_{\langle k\rangle}) \cap \mathrm{ann}(\hat{f}_{\langle k\rangle}) & \mathrm{ann}(\hat{f}_{\langle k\rangle}) \cap \mathrm{ann}(\hat{g}_{\langle k\rangle}) \end{bmatrix},$$

it follows from Theorem 1.2 that $M_2(\langle \widehat{(kd^{-1})}_{\langle k\rangle}\rangle) \subseteq \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle})$. Therefore each equivalence class in $(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}))^*$ has $|\langle \widehat{(kd^{-1})}_{\langle k\rangle}\rangle|^4$ elements.
We define addition $\boxplus$ and multiplication $\boxdot$ on $(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}))^*$ by

$$\widehat{A}^*_{\langle k\rangle} \boxplus \widehat{C}^*_{\langle k\rangle} = (\widehat{A}_{\langle k\rangle} + \widehat{C}_{\langle k\rangle})^* \quad \text{and} \quad \widehat{A}^*_{\langle k\rangle} \boxdot \widehat{C}^*_{\langle k\rangle} = (\widehat{A}_{\langle k\rangle} \boxdot \widehat{C}_{\langle k\rangle})^*. \tag{10}$$

It is easy to show that the binary operations $\boxplus$ and $\boxdot$ are well-defined and that the triple $\langle (\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}))^*, \boxplus, \boxdot\rangle$ is a ring, which we sometimes, if the context is clear, denote by $(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}))^*$.

Using the following well-known result, Corollary 3.3 can easily be proved.

Theorem 3.2. *If $A_1, \ldots, A_m$ are ideals in a ring S (not necessarily commutative or with a unit), then there is a monomorphism of rings $\phi : S/(A_1 \cap \cdots \cap A_m) \to S/A_1 \oplus \cdots \oplus S/A_m$ defined by $\phi(s + (A_1 \cap \cdots \cap A_m)) = (s + A_1, \ldots, s + A_m)$. If $S^2 + A_i = S$ for all i and $A_i + A_j = S$ for all $i \neq j$, then ϕ is an isomorphism of rings.*

Corollary 3.3. *Let $R/\langle k\rangle$ be finite, and let $k = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$, with $p_1, \ldots, p_m$ different primes and $n_1, \ldots, n_m \geq 1$. Then*

(i) $\phi : R/\langle k\rangle \to R/\langle p_1^{n_1}\rangle \oplus R/\langle p_2^{n_2}\rangle \oplus \cdots \oplus R/\langle p_m^{n_m}\rangle$ *defined by*

$$\phi(\hat{r}) = (\theta_{\langle p_1^{n_1}\rangle}(r), \theta_{\langle p_2^{n_2}\rangle}(r), \cdots, \theta_{\langle p_m^{n_m}\rangle}(r))$$

is an isomorphism.

(ii) $\Phi : M_2(R/\langle k\rangle) \to M_2(R/\langle p_1^{n_1}\rangle) \oplus \cdots \oplus M_2(R/\langle p_m^{n_m}\rangle)$ *defined by*

$$\Phi([\hat{b}_{ij}]) = (\Theta_{\langle p_1^{n_1}\rangle}([b_{ij}]), \ldots, \Theta_{\langle p_m^{n_m}\rangle}([b_{ij}]))$$

is an isomorphism.

We need the following trivial results in the next Lemma 3.6.

Lemma 3.4. *Let $S, S_1, \ldots, S_m$ be rings, $s \in S$ and let $\Gamma : S \to S_1 \oplus \cdots \oplus S_m$ defined by $\Gamma(s) = (s_1, \ldots, s_m)$ be an isomorphism. Then $t \in \mathrm{Cen}_S(s)$ if and only if $t_i \in \mathrm{Cen}_{S_i}(s_i)$, for all i.*

Lemma 3.5. *Let $R/\langle k\rangle$ be finite. An element $\hat{b} \in R/\langle k\rangle$ is invertible if and only if $\gcd(b, k) = 1$.*

Lemma 3.6. *Let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$ and let $k \in R$. If $\gcd(e-h, f, g, k)$ $= 1$, then*

$$|\mathrm{Cen}(\widehat{B}_{\langle k\rangle})| = |R/\langle k\rangle|^2.$$

Proof. Suppose $k = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$, where $p_1, \ldots, p_m$ are different primes and $n_i \geq 1$ for all i. It follows from Lemma 3.3(ii) and Lemma 3.4 that

$$\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}) \cong \bigoplus_{i=1}^{m} \mathrm{Cen}_{M_2(R/\langle p_i^{n_i}\rangle)}(\widehat{B}_{\langle p_i^{n_i}\rangle}).$$

Therefore,

$$|\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle})| = \prod_{i=1}^{m} |\mathrm{Cen}_{M_2(R/\langle p_i^{n_i}\rangle)}(\widehat{B}_{\langle p_i^{n_i}\rangle})|.$$

If we can show that $|\mathrm{Cen}_{M_2(R/\langle p_i^{n_i}\rangle)}(\widehat{B}_{\langle p_i^{n_i}\rangle})| = |R/\langle p_i^{n_i}\rangle|^2$, for all i, it follows, again from Lemma 3.3(ii) and Lemma 3.4, that

$$|\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle})| = \prod_{i=1}^{m} |R/\langle p_i^{n_i}\rangle|^2 = |R/\langle k\rangle|^2.$$

Let p_i be an arbitrary prime in the prime factorization of k. Since $\gcd(e - h, f, g, k) = 1$, it follows that $p_i \nmid f$ or $p_i \nmid g$ or $p_i \nmid e - h$. Thus, by Lemma 3.5, at least one of $\hat{f}_{\langle p_i^{n_i}\rangle}$, $\hat{g}_{\langle p_i^{n_i}\rangle}$ or $\hat{e}_{\langle p_i^{n_i}\rangle} - \hat{h}_{\langle p_i^{n_i}\rangle}$ is invertible in $R/\langle p_i^{n_i}\rangle$.

If $\hat{f}$ is invertible in $R/\langle p_i^{n_i}\rangle$ with inverse $\hat{t}$, say, then given that $\gcd(e-h, f, g, p_i^{n_i}) = 1$, it follows from Corollary 2.11 and Lemma 2.2(ii) that

$$\begin{aligned} \mathrm{Cen}_{M_2(R/\langle p_i^{n_i}\rangle)}(\widehat{B}) &= \mathrm{Cen}\left(\begin{bmatrix} \hat{e} & \hat{f} \\ \hat{g} & \hat{h} \end{bmatrix}\right) = \mathrm{Cen}\left(\begin{bmatrix} \hat{t}\hat{e} & \hat{1} \\ \hat{t}\hat{g} & \hat{t}\hat{h} \end{bmatrix}\right) \\ &= \left\{ \hat{a}\begin{bmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{bmatrix} + \hat{b}\begin{bmatrix} \hat{0} & \hat{1} \\ \hat{t}\hat{g} & -\hat{t}(\hat{e}-\hat{h}) \end{bmatrix} \,\middle|\, \hat{a}, \hat{b} \in R/\langle p_i^{n_i}\rangle \right\}. \end{aligned} \tag{11}$$

It can be similarly shown that if $\hat{g}$ is invertible in $R/\langle p_i^{n_i}\rangle$ with inverse $\hat{t}$, say, then

$$\mathrm{Cen}_{M_2(R/\langle p_i^{n_i}\rangle)}(\widehat{B}) = \left\{ \hat{a}\begin{bmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{bmatrix} + \hat{b}\begin{bmatrix} \hat{0} & \hat{t}\hat{f} \\ \hat{1} & -\hat{t}(\hat{e}-\hat{h}) \end{bmatrix} \,\middle|\, \hat{a}, \hat{b} \in R/\langle p_i^{n_i}\rangle \right\}; \tag{12}$$

and if $\hat{e}-\hat{h}$ is invertible in $R/\langle p_i^{n_i}\rangle$ with inverse $\hat{t}$, say, then

$$\mathrm{Cen}_{M_2(R/\langle p_i^{n_i}\rangle)}(\widehat{B}) = \left\{ \hat{a}\begin{bmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{bmatrix} + \hat{b}\begin{bmatrix} \hat{0} & -\hat{t}\hat{f} \\ -\hat{t}\hat{g} & \hat{1} \end{bmatrix} \,\middle|\, \hat{a}, \hat{b} \in R/\langle p_i^{n_i}\rangle \right\}. \tag{13}$$

It is easy to see that the number of elements in the sets in (11), (12) and (13) are $|R/\langle p_i^{n_i}\rangle|^2$.

$\square$

Lemma 3.7. *Let $k \in R$, let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R)$ and let*

$$B' = \begin{bmatrix} d^{-1}(e-h) & d^{-1}f \\ d^{-1}g & 0 \end{bmatrix},$$

where $d := \gcd(e-h, f, g, k)$, then

$$(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}))^* \cong \mathrm{Cen}_{M_2(R/\langle kd^{-1}\rangle)}(\widehat{B'}_{\langle kd^{-1}\rangle}).$$

Proof. Since

$$\begin{aligned} &\widehat{A}^*_{\langle k\rangle} \in (\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}))^* \Leftrightarrow \widehat{A}_{\langle k\rangle} \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}) \\ \Leftrightarrow\ &\widehat{A}_{\langle k\rangle} \in \mathrm{Cen}_{M_2(R/\langle k\rangle)}\left(\begin{bmatrix} \hat{e}_{\langle k\rangle} - \hat{h}_{\langle k\rangle} & \hat{f}_{\langle k\rangle} \\ \hat{g}_{\langle k\rangle} & \hat{0}_{\langle k\rangle} \end{bmatrix}\right) \\ \Leftrightarrow\ &A\begin{bmatrix} e-h & f \\ g & 0 \end{bmatrix} - \begin{bmatrix} e-h & f \\ g & 0 \end{bmatrix}A \in M_2(\langle k\rangle) \\ \Leftrightarrow\ &AB' - B'A \in M_2(\langle kd^{-1}\rangle) \Leftrightarrow \widehat{A}_{\langle kd^{-1}\rangle} \in \mathrm{Cen}_{M_2(R/\langle kd^{-1}\rangle)}(\widehat{B'}_{\langle kd^{-1}\rangle}) \end{aligned}$$

and

$$\begin{aligned} \widehat{A}^*_{\langle k\rangle} = \widehat{C}^*_{\langle k\rangle} &\Leftrightarrow \widehat{A}_{\langle k\rangle} - \widehat{C}_{\langle k\rangle} \in M_2(\langle \widehat{kd^{-1}}_{\langle k\rangle}\rangle) \\ &\Leftrightarrow A - C \in M_2(\langle kd^{-1}\rangle) \Leftrightarrow \widehat{A}_{\langle kd^{-1}\rangle} = \widehat{C}_{\langle kd^{-1}\rangle}. \end{aligned}$$

it follows that $\Gamma : (\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}))^* \to \mathrm{Cen}_{M_2(R/\langle kd^{-1}\rangle)}(\widehat{B'}_{\langle kd^{-1}\rangle})$, defined by

$$\Gamma(\widehat{A}^*) = \widehat{A}_{\langle kd^{-1}\rangle},$$

is a well-defined function which is $1-1$ and onto. It can be easily shown that Γ is a homomorphism. $\square$

We are finally able to determine the number of elements in the centralizer of a matrix in $M_2(R/\langle k\rangle)$, if R is a UFD and $R/\langle k\rangle$ is finite.

Theorem 3.8. *Suppose R is a UFD, $k \in R$ is a nonzero nonunit such that $R/\langle k\rangle$ is finite, and*

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_2(R),$$

then

$$|\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle})| = |R/\langle kd^{-1}\rangle|^2 \cdot |\langle(\widehat{kd^{-1}})_{\langle k\rangle}\rangle|^4,$$

where d :=gcd$(e-h, f, g, k)$.

Proof. With B' as in Lemma 3.7, it follows from Lemma 3.6 that

$$|\mathrm{Cen}_{M_2(R/\langle kd^{-1}\rangle)}(\widehat{B'}_{\langle kd^{-1}\rangle})| = |R/\langle kd^{-1}\rangle|^2.$$

Since each equivalence class in $(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}))^*$ has cardinality $|\langle(\widehat{kd^{-1}})_{\langle k\rangle}\rangle|^4$, it follows that

$$|\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle})| \quad = \quad |(\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle}))^*||\langle(\widehat{kd^{-1}})_{\langle k\rangle}\rangle|^4,$$

and so Lemma 3.7 implies that

$$\begin{aligned} |\mathrm{Cen}_{M_2(R/\langle k\rangle)}(\widehat{B}_{\langle k\rangle})| &= |\mathrm{Cen}_{M_2(R/\langle kd^{-1}\rangle)}(\widehat{B'}_{\langle kd^{-1}\rangle})||\langle(\widehat{kd^{-1}})_{\langle k\rangle}\rangle|^4 \\ &= |R/\langle kd^{-1}\rangle|^2|\langle(\widehat{kd^{-1}})_{\langle k\rangle}\rangle|^4. \end{aligned}$$

$\square$

Example 3.9. Let $R = \mathbb{Z}[i]$, $k = 12$ so that $R/\langle k\rangle = \mathbb{Z}_{12}[i]$ (see [**5**], p. 604, Theorem 1) and let

$$\widehat{B} = \begin{bmatrix} \widehat{4i} & \hat{3}+\widehat{6i} \\ \widehat{9i} & \hat{i} \end{bmatrix}.$$

Using the fact that every matrix is a $\langle k\rangle$-matrix if R is a PID, note that, according to Lemma 2.2(ii) and Theorem 1.2

$$\begin{aligned} \mathrm{Cen}_{M_2(\mathbb{Z}_{12}[i])}(\widehat{B}_{\langle 12\rangle}) &= \Theta_{\langle 12\rangle}\left(\left\{\begin{bmatrix} a & (1+2i)b \\ 3ib & a-3ib \end{bmatrix}\middle| a,b \in \mathbb{Z}[i]\right\}\right) + \begin{bmatrix} \langle\hat{4}\rangle & \langle\hat{4}\rangle \\ \langle\hat{4}\rangle & \hat{0} \end{bmatrix} \\ &= \left\{\begin{bmatrix} \hat{a}+\widehat{4c} & (\hat{1}+\widehat{2i})\hat{b}+\widehat{4m} \\ \widehat{3ib}+\widehat{4n} & \hat{a}-\widehat{3ib} \end{bmatrix}\middle| \hat{a},\hat{b},\hat{c},\hat{m},\hat{n} \in \mathbb{Z}_{12}[i]\right\}. \end{aligned}$$

Now, since $\gcd(3i, 3+6i, 9i, 12) = 3$, let $d = 3$ so that $kd^{-1} = 12 \cdot 3^{-1} = 4$. Since

$$|\mathbb{Z}[i]/\langle 4\rangle| = |\{a + ib \mid a, b \in \mathbb{Z}_4\}| = 16 \qquad \text{and} \qquad |\langle \hat{4}_{\langle 12\rangle}\rangle| = 9$$

it follows from Theorem 3.8 that

$$|\mathrm{Cen}_{M_2(\mathbb{Z}_{12}[i])}(\widehat{B}_{\langle 12\rangle})| = 16^2 \cdot 9^4 = 1679616.$$

For 2×2 matrices over a factor ring of $\mathbb{Z}$ we have the following result.

Corollary 3.10. *Let* $\widehat{B} = \begin{bmatrix} \hat{e} & \hat{f} \\ \hat{g} & \hat{h} \end{bmatrix} \in M_2(\mathbb{Z}_k)$, *then* $|\mathrm{Cen}(\widehat{B})| = (kd)^2$, *where* $d = \gcd(e - h, f, g, k)$.

Proof. According to Theorem 3.8

$$|\mathrm{Cen}_{M_2(\mathbb{Z}_k)}(\widehat{B}_{\langle k\rangle})| = |\mathbb{Z}_{kd^{-1}}|^2 |\langle(\widehat{kd^{-1}})_{\langle k\rangle}\rangle|^4 = (kd^{-1})^2 d^4 = (kd)^2.$$

□

Example 3.11. Let $\widehat{B}_{\langle 12\rangle} = \begin{bmatrix} \hat{2}_{\langle 12\rangle} & \hat{2}_{\langle 12\rangle} \\ \hat{4}_{\langle 12\rangle} & \hat{8}_{\langle 12\rangle} \end{bmatrix}$. Since $\gcd(6, 2, 4, 12) = 2$, it follows that

$$|\mathrm{Cen}_{M_2(\mathbb{Z}_{12})}(\widehat{B}_{\langle 12\rangle})| = (12 \cdot 2)^2 = 24^2 = 576.$$

Remark 3.12. A natural example to include in this section, if such an example exists, would be one of a UFD R, which is not a PID, and a nonzero nonunit $k \in R$, such that $R/\langle k\rangle$ is finite. Unfortunately we could not find such an example. Neither have we been able to prove that if R is UFD and $k \in R$ is a nonzero nonunit such that $R/\langle k\rangle$ is finite, then R is a PID.

References

[1] E.R. Berlekamp, Distribution of cyclic matrices in a finite field, *Duke Math. J.* **33** (1966), 45–48.

[2] D. Bollman and H. Ramírez, On the number of nilpotent matrices over $\mathbb{Z}_m$, *Reine Angew. Math.* **238** (1969), 85–88.

[3] D. Bollman and H. Ramírez, On the enumeration of matrices over finite commutative rings, *Amer. Math. Monthly* **76** (1969), 1019–1023.

[4] L. Carlitz, Representations by quadratic forms in a finite field, *Duke Math. J.* **21** (1954), 123–137.

[5] G. Dresden and W.M. Dymàček, Finding factors of factor rings over the Gaussian integers, *Amer. Math. Monthly* **112**(7) (2005), 602–611.

[6] P.R. Halmos, *Finite-dimensional vector spaces*, 2nd edition, Van Nostrand Company, Princeton, N.J., 1958.

[7] X. Hou, On the centralizer of the centralizer of a matrix, *Linear Algebra Appl.* **256** (1997), 251–261.

[8] T.W. Hungerford, *Algebra*, Holt, Rinehart and Winston, New York, London, 1974.

[9] N. Jacobson, *Lectures in abstract algebra*, Vol. II, Van Nostrand Company, Princeton, N.J., 1953.

[10] M.S. Marais, *The Centralizer of an I-matrix in $M_2(R/I)$, R a UFD*, Algebra Colloquium, to appear.

[11] G.C. Nelson and T. Ton-That, Multiplicatively closed bases for $C(A)$, *Note Mat.* **26**(2) (2006), 81–104.

[12] I. Reiner, On the number of matrices with given characteristic polynomial, *Illinois J. Math.* **5** (1961), 324–329.

[13] D.A. Suprunenko and R.I. Tyshkevich, *Commutative matrices*, Academic Press, New York, 1968.

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